Perspectives in mesoscopic superconductivity: Magneto-electric effects, Topological Majorana modes, and beyond

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The topological perspectives in mesoscopic superconductivity attract specific attention at the moment, due to the possible immediate realization of emergent Majorana modes in heterostructures. The topological sector of superconductivity requires the understanding of strong spin-orbit and spin-splitting interactions, in addition to impurities and defects inherent to its practical realisation in heterostructures. This project aims at developing topological methods into the statistical field theory of superconductivity. In particular, a quantum transport method able to deal with impurities and interfaces problems will be developed, describing non-trivial topology of superconductivity. In addition, thermal Hall effects and their quantum counterparts will be studied. Finally, the magneto-electric phenomenology combining spin texture and superconductivity will be explored beyond the low-energy topological sector, in order to understand how the neutral and spin-less Majorana quasi-particle emerges and dominates from the combination of spin and charge correlations.

Keywords: mesoscopic superconductivity ; topological superconductivity ; Majorana mode ; magneto-electric effect ; gauge theory ; spin-splitting ; spin-orbit ; statistical field theory ; quasi-classic Green's functions ; non-Abelian gauge-covariant transport formalism ; deformation quantization

Contents

I. INTRODUCTION

Condensed matter and the technology of information have always gone hand to hand, as illustrated with e.g. the transistor, logic circuits, ... Over decades, the miniaturization of circuits allowed to focus on the microscopic capacities of the electronic devices, which lead to the emergence of mesoscopic physics on the one side (Nazarov and Blanter, 2009), and the quest for a quantum computer on the other side (Nielsen and Chuang, 2000). A decade ago, a new scheme of quantum computation has been proposed, called the topological quantum computation scheme (Kitaev, 2003). In this scheme, one is looking for exotic states of matter able to resist to the interactions with the uncontrollable environment. This protection emerges from the non-local properties of the exchange of elementary excitations. Beyond the well known fermionic and bosonic exchange corresponding to an accumulated phase of π and 0 respectively, two exotic particles can acquire intermediary phase during their interchange[1]. Anyons are examples of such exotic states, and might serve as the elementary particles encoding quantum information in a robust way (Nayak *et al.*, 2008; Stern and Lindner, 2013).

At a practical level, there are immediate perspectives in the search for Majorana modes (Alicea, 2012; Beenakker, 2013). Despite this emergent particle can not be used for universal quantum computation, its properties under exchange of two modes (called braiding) are still of interest for encoding quantum memory in a robust way. In addition the Majorana modes are one of the elementary member of the anyon family (they are Abelian anyons), and as such they might be the simplest emergent particle with topological properties that can be studied. So one can see the Majorana modes as some toy model for more complicated situations not yet discovered.

Majorana modes are predicted to exist in low dimensional and spin-less p-wave superconductors, when the particle-hole symmetry is the only surviving one. In p-wave superconductors, the Cooper pairs appear with a odd-momentum state, allowing equal spin polarization, and eventually the emergence of a spin-less condensate (Volovik, 2003). In addition, the excitation quasiparticles of a superconductor (called Bogoliubov quasi-particles) are superposition of electrons and holes. Under peculiar conditions (mainly at zero energy in spin-less superconductors), some Bogoliubov quasi-particles γ turn out to be self-adjoint $\gamma^{\dagger} = \gamma$, the definition of a Majorana quasi-particles (Majorana, 1937). Actually, these modes manifest as unpaired[2] quasi-particles (called Majorana modes) localized at the edges of the p-wave superconductor, because they only exist inside the superconducting gap (Kitaev, 2001). As any Majorana particle, they can not be charged ; thus the emerging Majorana modes are both spin-less and charge-less quasi-particles, making their detection quite challenging.

Unfortunately, there is no known example of p-wave superconductors in real materials[3]. To generate unpaired Majorana modes, one can alternatively combine different properties of magnetic and superconducting materials. For instance, a conventional superconductor

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[1] Eventually, this *phase* is in fact a rotation between internal degrees of freedom of the particles, as in the case of non-abelian anyons.

[2] Any fermionic degree of freedom with annihilation c and creation $c^{\dagger}$ operators can be split into two Majorana modes γ_1 and γ_2 as $c = (\gamma_1 + \mathbf{i}\gamma_2)/\sqrt{2}$ and $c^{\dagger} = (\gamma_1 - \mathbf{i}\gamma_2)/\sqrt{2}$ verifying $\gamma_{1,2} = \gamma_{1,2}^{\dagger}$. The topological protection comes from the possibility to separate these two sub-modes in space, for instance at the edges of a p-wave superconducting wire (Kitaev, 2001).

[3] There are p-wave states in superfluids, though, see (Leggett, 1975) or (Volovik, 2003).

(s-wave) with spin-orbit coupling has no definite spin and momentum states, and can thus present a superposition of even- and odd-in-momentum Cooper correlations, the so-called s/p-mixing when helicity is the only conserved quantity (Gor'kov and Rashba, 2001). In addition a Zeeman effect splits the two helicity bands in energy. If the spin-splitting is strong enough, the low energy sector is then described by a helical fluid, in which the superconductivity is an emergent spin-less p-wave one (Alicea *et al.*, 2011), see Fig.1. Thus mixing strong spin-orbit and spin-splitting effects lead to an effective topological superconductor in which unpaired Majorana modes appear. A strong spin-splitting effect, which tends to polarize all the electrons along a given direction, is nevertheless detrimental to conventional superconductivity composed of Cooper pairs with opposite spin directions. However, a robust conventional s-wave Cooper pairing and a strong Zeeman spin-splitting can coexist in heterostructures, when the superconducting properties are shared with a non-superconducting metal put in contact with a superconductor.

Then the generation of Majorana modes in realistic situation imposes to consider the phenomenology of heterostructures when a superconductor is put in contact with a semiconductor exhibiting strong spin-orbit and spin-splitting effects. These systems present inherent interfaces, impurities, intrinsic disorder, ... problems, in addition to the competition between proximity induced superconductivity and spin textures.

Other proposals predict the topological degree of freedom in heterostructures to emerge when a topological insulator is contacted with a superconductor under Zeeman interaction (Fu and Kane, 2008), or spin-ordered ad-atoms deposited on top of a superconductor (Nadj-Perge *et al.*, 2013). In all situations the unpaired Majorana modes appear because of the combination of spin texture and superconductivity.

Despite a few experiments seem promising in their quest for the Majorana modes (Deng *et al.*, 2012; Mourik *et al.*, 2012; Nadj-Perge *et al.*, 2014), one has to realize that the phenomenology of the interaction between superconductivity and spin texture is far from being

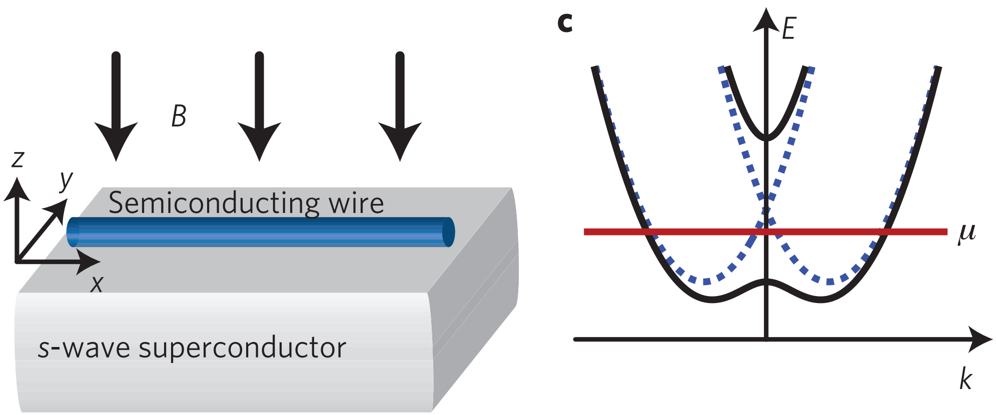

Figure 1 Right panel: Energy versus momentum dispersion relation of an ideal semiconductor when both spin-orbit and spin-splitting effects are present. The usual parabola degenerate in spin is split due to the spin-orbit effect towards the dashed parabolas, corresponding to helical dispersion (i.e. spin-momentum locking). Then a strong spin-splitting (Zeeman) effect separates these two helical bands, resulting in the plain curves. When the chemical potential (red line) is between the two separated bands, the low energy sector coorresponds to a helical fluid. When such a system is put in contact with a superconductor (left panel), Majorana modes can emerge. Picture from (Alicea *et al.*, 2011).

completely understood. In fact, Majorana mode would be only one effect associated with this rich phenomenology.

The ultimate goal of this project is to encapsulate the full phenomenology associated with the combination of spin-orbit, spin-splitting and superconductivity within a few theoretical concepts. Beyond the exploration of the presence of the Majorana modes in superconducting heterostructures, one aims at clarifying how this spin-less and charge-less degree of freedom will emerge from the spin and charge interactions present in superconducting heterostructures. I would like to address the following questions

- when does the presence of Majorana modes can be certified ?
- when does the Majorana phenomenology dominate any other phenomenology associated with spin-orbit and spin-splitting effects in superconducting bulk and heterostructure ?

These questions require to understand first the effect of spin texture on superconductivity, especially in heterostructures. Then one has to study how the topological sector emerges from the combination of spin-texture and superconductivity in generic term (i.e. in the presence of interfaces, impurities, and so on). Finally, one has to understand under which circumstances the Majorana phenomenology will dominate the other ones, in order to interpret the recent experiments correctly and to propose and design new experiments.

The present project is quite challenging, and more certainly cannot be successful without a powerful enough theoretical model able to encapsulate many different effects associated with spin-orbit, spin-splitting interactions in the presence of superconducting correlations, in addition to impurities, disorder, interfaces, temperatures conditions in heterostructures. In fact such a powerful methodology already exists in the form of a gauge-covariant transport formalism, when charge and spin degrees of freedom are treated on equal footing (Konschelle, 2014). The presentation of such a method will be given in section II. Then a detailed presentation of the project will be proposed in section III. I will briefly sketch in section IV some other interesting questions associated with either more exotic states with applications in quantum computation, or fundamental problems in theoretical physics that the transport formalism may help to resolve.

II. STATE OF THE ART IN MESOSCOPIC SUPERCONDUCTIVITY

Mesoscopic physics is the study of medium sized devices when quantum effects start to play a significant role in term of the electronic transport[4]. In particular, the coherence length, the characteristic length over which quantum effect are present, must be larger than the characteristic length of the device. When discussing superconductivity, this coherence length is pretty large (up to 100nm - 1μm), and so mesoscopic superconductivity signifies that one considers heterostructures, when proximity effect plays a significant role. That way, one can mix the property of a normal system (for instance its spin texture) with the superconducting correlations (De Gennes, 1999).

[4] Mesoscopic physics in fact also discusses the response to optical excitations, but I will focus on the electronic transport for this presentation. For more details about the mesoscopic photonics, see (Akkermans and Montambaux, 2004).

In order to study mesoscopic superconductivity, there are several theoretical methods at hands. The so-called Bogoliubov-deGennes (BdG) equations are commonly used for instance. They consists in an elegant generalization of the Schrödinger equation in an enlarged space taking into account the particle-hole symmetry (the Nambu space), and they describe the quasi-particles dynamics (i.e. the dynamics of the Bogoliubov quasi-particles), see e.g. (De Gennes, 1999). In addition, the understanding of spin-orbit and spin-splitting effects can be done in a straightforward way using the BdG methods. As such, they have been thoroughly used in the description of the topological sector of superconducting heterostructures (Beenakker, 2013). Nevertheless, the BdG formalism is well adapted to ballistic systems at zero temperature only, and the formalism becomes quite cumbersome when one tries to understand realistic situations when disordered interfaces, impurities and temperature effects are dominant in the heterostructures.

An other methodology has been developed over the years in order to tackle more realistic situations. It consists in a quantum field theory adapted to the statistical mechanics (Abrikosov *et al.*, 1963). This formalism was generalized to describe the superconducting correlations at any temperatures (Gor'kov, 1958, 1959), and the problem of impurities in superconducting alloys, either magnetic (Abrikosov and Gor'kov, 1961) or not (Abrikosov and Gor'kov, 1959). In addition, a simplified version of this statistical field theory (SFT) has been proposed, the so-called quasi-classic methods (Eilenberger, 1968; Larkin and Ovchinnikov, 1969). This latter consists in a low energy expansion of the SFT à la Wigner, i.e. when transposing the SFT to the phase space and keeping only the lowest quantum correction to the classical physics (Moyal, 1947; Polkovnikov, 2010; Wigner, 1932). The resulting method, called the transport method (Kadanoff and Baym, 1962; Rammer and Smith, 1986), has proven really effective over the years, especially when discussing inhomogeneous problems of superconductivity, as the problem of vortices and heterostructures (Kopnin, 2001; Langenberg and Larkin, 1986; Wilhelm *et al.*, 1999). In addition the transport formalism allows a simple description of either the ballistic or diffusive limit, some perturbative approaches in the crossover regime, and some convenient parameterization of interfaces in heterostructures. It can also be generalized in a straightforward way to time-dependent problem (Langenberg and Larkin, 1986), though one often has to rely on numerical computation in this case. The quasi-classic formalism has been adapted to superconducting heterostructures with ferromagnetic order as well (Bergeret *et al.*, 2005; Buzdin, 2005).

So far, the transport formalism has not been generalized to the topological problems. One of the difficulties is that it describes the electron excitations in superconductors, and not the quasi-particles excitations as the BdG formalism. For instance, for a Majorana mode, it seems simpler to prove that a self-adjoint Bogoliubov quasi-particle is an eigenmode of a topological superconductor, than studying this topological degree of freedom using electron correlation functions like the Green's functions, in which case the Majorana mode manifests itself as a zero energy bound state given by the superposition of the normal and anomalous Green-Gor'kov functions. An other difficulty was to generalize quasi-classic methods to complicated band structures, as e.g. when strong spin-orbit and strong spin-splitting effects combine in a superconducting element.

Thus the methods of mesoscopic superconductivity seem to be somehow schizophrenic when applied to the topological sector:

- the Bogoliubov-deGennes formalism can be straightforwardly generalized to this sector, but it is limited to clean systems at zero-temperature

- the quasi-classic formalism describes efficiently the microscopic details of impurities, interfaces and temperature, but seems quite inconvenient when dealing with the topological sector

In addition the straightforward way of including spin-orbit and spin-splitting effects in superconductors using the transport formalism is quite cumbersome. Then during years, one have had no access to the topological sector of superconductors using the quasi-classic methods. The situation changed a few years ago, when I start developing the so-called non-Abelian gauge-covariant transport method adapted for superconductivity (Konschelle, 2014).

The non-Abelian gauge-covariant transport formalism (NAT) takes into account both the charge and spin degrees of freedom on the same footing. It is for instance well known that the Maxwell electromagnetism can be accounted for as a gauge degree of freedom, when one associates an Abelian group redundancy to the electronic wave-function (Aitchison and Hey, 2004). It is perhaps less known that the electronic spin can be described in the same way, though using a non-Abelian group redundancy instead of an Abelian one[5]. This is especially useful when discussing spin-orbit effect in condensed matter problems (Berche and Medina, 2013; Fröhlich and Studer, 1992). Then one can take this gauge redundancy as a prescription to describe the electron spin using a gauge-covariant formalism, and use this prescription as a guiding principle toward the establishment of the transport formalism. This kind of transport formalism has been developed in the high-energy sector in order to describe the quark-gluon plasma (Elze and Heinz, 1989) ; though there have been complications due to the confinement problem. In contrary, in the low-energy sector, one can successfully apply the formalism in order to describe complicated situations, both in normal (Gorini *et al.*, 2010) and superconducting (Konschelle, 2014) metals.

So far, the NAT has been applied to peculiar situations without topological effect. Namely, we studied the anomalous phase shift in Josephson junction (called a φ_0-junction) when spin-orbit and spin-splitting effects conspire to generate a spin-current inside the normal region. This spin-current is then converted to a charge current by an inverse spin-Hall-like effect (called an inverse Edelstein effect, or spin-galvanic effect). This charge current is responsible for the anomalous phase-shift in the Josephson system, or equivalently to the generation of a spontaneous super-current (Konschelle *et al.*, 2015). Really recently, this phenomenology was experimentally studied successfully (Szombati *et al.*, 2015). Actually, we are exploring the spin-current versus phase-difference relation in Josephson junctions, and the generalization of the current-phase relations when any kind of magnetic texture is present in the junction. The results we obtained nicely generalize the literature on superconducting-ferromagnet-superconducting Josephson junction since a few decades, and generalize them to systems with spin texture [6].

The aim of the project is to continue investigating the magneto-electric effects in superconducting heterostructures under strong spin-orbit and spin-splitting effects, and in addition to generalize the transport formalism to incorporate topological effects. Then one will have a working theory able to distinguish clearly between the Majorana modes and the rich phenomenology associated to spin-texture in superconductors.

[5] Namely, the electron spin is described using an SU(2) gauge, whereas the electron charge can be described using the simpler U(1) formalism.

[6] F. Konschelle, I.V. Tokatly and F.S. Bergeret. *In preparation.* (2016)

III. PERSPECTIVES IN MESOSCOPIC SUPERCONDUCTIVITY: MAGNETO-ELECTRIC EFFECTS, TOPOLOGICAL MAJORANA MODES AND BEYOND

Despite the non-Abelian gauge-covariant transport formalism (NAT) seems really promising in its description of superconducting correlations in materials with both spin-orbit and spin-splitting effects, it is still at its infancy (see section II). There are several ground breaking generalizations to do, in particular in the possibilities to describe topological properties of superconducting heterostructures. Below I propose a few perspectives along this line. The ultimate goal of this project is to understand how the Majorana modes (or eventually more exotic topological quasi-particles at a longer time scale, see section IV) emerge in superconducting systems when spin-orbit and spin-splitting interactions mix charge and spin degrees of freedom.

A. Effective quantum field theory of topological superconductor

A powerful tool to describe the low energy excitations of a material consists in establishing an effective theory. For topological materials, the low energy phenomenology is well described by some topological quantum field theories (Qi, 2013). They usually consists in gauge-covariant (not invariant) terms permitted by boundary conditions, or symmetry of the partition function[7]. So far, there have been systematic procedures developed to verify whether the low energy phenomenology of some normal systems is described in terms of such topological actions, but these procedures are difficult to generalize to superconductors (Dunne, 1998; Qi, 2013). In contrary, a gradient expansion has been proved helpful for superfluid systems (Volovik, 1988; Volovik and Yakovenko, 1989). This last method uses the same tool as the transport formalism, since the gradient expansion (also called Moyal expansion) is common in both situations.

Nevertheless, superfluids and superconductors require different methodologies. On the one hand, superconductors are intrinsically charged, and so the interactions between Cooper pairs and the electromagnetic field have to be properly taken into account. This can be done in a straightforward way using the NAT which already takes into account spin and charge on equal footing. On the other hand, though, superconductors lay on materials, and the effects of lattice and band structure have to be accounted for as well. Of utmost importance for topological issues is the description of non-trivial curvature of the band structure, known as the Berry curvature (Xiao *et al.*, 2009) and associated to topological effects in normal systems (Hasan and Kane, 2010; Qi and Zhang, 2011). At the present time, the NAT is not completely generalized to take into account non trivial band topology. In particular, it is not yet clear how the presence of non-trivial Berry curvature will affect the transport equations. However, recent progress made for normal systems, have shown how to include anomalies and non-trivial Berry phenomenology using the quasi-classic methods (Son and Yamamoto, 2012, 2013; Sundaram and Niu, 1999; Wickles and Belzig, 2013; Wong and Tserkovnyak, 2011). These progress are really encouraging, and call for further generalizations in the case of topological superconductivity. In that case, one expects that the spectrum of the quasi-particles can become non-trivial, a close analogy with the band spectrum in normal

[7] Said quickly, the action S may depend on the gauge choice, but the partition function $Z = e^{\mathrm{i}S}$ will not for some topological action S.

metals and semi-conductors.

I thus propose to generalize known methods for superfluid and normal systems to superconducting ones, where the non-trivial topology of the spectrum of quasi-particles excitations can be accounted for in the same way as the band structure in normal metal or semiconductors. One will especially focus on the possibility to generate topological actions in superconductors with spin texture. This will permit a classification of the topological phenomenology of superconductors in term of a few concepts, like the gauge potentials and fields (for charge, spin and spectrum/band structure now). In addition, one will understand the necessary and sufficient ingredients required for the emergence of a topological sector of the theory.

Once equipped with this methods, one can understand in more details how the higher energy contributions will alter the pure topological terms. That way, we will understand the competition between the topological phenomenology, the unconventional superconductivity effects (in particular p-wave superconductivity) and the usual ones (say the s-wave superconductivity for short). Nonetheless this is of interest for experimental characterization of new materials, but it provides methods for fundamental research on exotic materials as well. In fact, since the transport formalism is best suited to discuss realistic experimental situations, one expects to make interesting breakthrough in the understanding of topological superconductors, both in bulk systems and in heterostructures. One will be able to help designing new experiments as well as interpreting accurately the actual ones. Moreover, since the NAT uses only a few concepts (namely the gauge potentials and fields) one will obtain the generic phenomenology of topological superconductivity in terms of these few concepts.

B. Thermal Hall effect in superconducting heterostructures

Being their own anti-particle, the Majorana quasi-particles can not be charged. In addition, they require the Kramers degeneracy to be violated in order to emerge (see section I). Thus the Majorana modes are neither charge nor spin detectable, which renders difficult their experimental evidence. At the present time, there is no proposed design of a smoking-gun experiment for the Majorana modes in superconducting heterostructures.

Despite they carry neither charge nor spin, the Majorana modes can still carry energy, and a thermal Hall effect has been predicted (Read and Green, 2000). This prediction lies on the analogy between the effect of a temperature gradient and the theoretical description of a gravitational field (Luttinger, 1964), which both appear as a scalar potential in the classical action. One eventually generalizes this analogy to the existence of a pseudo-magnetic field describing some kind of temperature vortices. It then appears an other formal analogy between the thermal effects and the electromagnetic ones, which are now both described in terms of scalar and vector potentials. In addition, the electromagnetic action describes the Hall effect, when the application of a magnetic field on flowing electrons generate a charge accumulation in the direction perpendicular to the flow. In analogy with the Hall effect, a heat flow can propagate perpendicular to the application of a temperature gradient, the so-called thermal Hall effect (or Righi-Leduc effect).

The prediction of the thermal Hall effect associated to Majorana modes follows the existence of a topological action (formally a gravitational anomaly) in the topologically non-trivial sector of the p-wave superconductivity, and its possible analogy to the fractional quantum Hall effect (Read and Green, 2000). Beyond the derivation of such a topological action from microscopic considerations (discussed in section III.A), one has to consider thermal effects in superconductors in order to describe the thermal Hall effect, and possibly its quantum counterpart. At the present time, NAT is not completely adapted to describe thermal phenomenology.

I thus propose to generalize the transport theory to include thermal effect, using Keldysh method. This will allow to describe the thermal transport in superconducting heterostructures with spin texture. Especially, one will be able to describe realistic situations, when interfaces and impurities are taken into account. The goal in this part of the project will be to efficiently describe experimental situations, in order to guide experimentalists in their quest for the Majorana modes. Beyond the description of the Majorana mode, the understanding of thermal phenomenology in superconductors with spin effects is of fundamental interest as well. One expects to make interesting discoveries, for instance in the field of caloritronic effects in superconductors, which are of practical interest at the mesoscopic level, either for cooling other circuits or to perform coherent manipulation of thermodynamic quantities (Giazotto *et al.*, 2006).

An other interesting possibility is to describe the so-called London momentum effect, when a superconductor put under displacement generates a magnetic field (Konschelle and Blanter, 2015; London, 1961), using quasi-classic formalism. Since the displacement of the materials can be seen as the action of a gravitational field, exactly as the thermal effect arises in effective theory, one expects a modification of the London momentum effect due to the presence of the Majorana modes which generate the thermal Hall effect. I thus propose to study the competition between the London momentum effect and the thermal Hall effect in topological superconductors. This would help designing a smoking gun experiment for the Majorana modes, which have no other degree of freedom that one can probe.

C. Magneto-electric effect in superconducting heterostructures

Despite the Majorana modes are of fundamental interests for condensed matter and quantum information, they correspond to one facet of the rich phenomenology associated with the combination of superconductivity and spin texture. One should explore the other facets of the combination of spin and superconducting correlations, in order to disentangle the neutral and spin-less Majorana mode from an unavoidable zoo of spin and charge phenomenologies. Indeed, Majorana quasi-particle corresponds to one specific mode in the spectrum of Bogoliubov quasi-particles (see sections I and II). All the other excitations can be charged, potentially spin-full, and might participate to some magneto-electric effects. These extra excitations are of potential interest for practical applications, as well as for the characterization of the Majorana phenomenology, yet to be completely established.

For instance, one can argue that the Majorana mode lies in the middle of the superconducting gap. Thus this topological mode is in principle isolated from the rest of the excitations, and can be distinguished from them. However, to be a zero energy mode inside the superconducting gap is far from being sufficient to characterize the Majorana mode, since there are many other possibilities for such a zero-energy excitation in proximity systems (Mourik *et al.*, 2012). In addition, these other zero-energy excitations are not necessarily of topological origin. One must thus eliminate these other possibilities, especially the ones detrimental for future applications in topological quantum computation. At the present time there is no generic method (either theoretical or experimental) to distinguish a topological mode from a non-topological one[8]. In addition a pedestrian approach, which would eliminate all the non-topological alternatives given equivalent responses than the topological ones, is unlikely to succeed. Nevertheless, the quasi-classic method is of considerable help to realize such an attempt. Indeed, it describes quite efficiently the dominant properties of a given superconductor with an economy of concepts and mathematical difficulties. Then it will become possible to disentangle the emerging Majorana phenomenology from the non-topological effects, and to characterize the situations when the Majorana phenomenology becomes the dominant response to some heterostructures. In addition the transport formalism offers unique perspectives in the study of emerging magneto-electric responses in superconducting devices.

This part of the project is devoted to the understanding of the phenomenology of spin-textured superconductors beyond the low energy sector where Majorana modes lie. One aims at describing in particular the magneto-electric effects that emerge from the combination of spin and charge phenomenologies, using the tool of the transport formalism. In addition to a the better characterization of the emerging Majorana mode, practical applications will range from heterostructures to bulk systems, and from the manipulation of the superconducting coherence via magnetic and/or electric fields to its reverse: the coherent manipulation of spin and charge textures. In particular, applications for a low energy spintronics are promising (see (Eschrig, 2015; Linder and Robinson, 2015) for reviews of the coherent spintronic before the spin-orbit coupling was under study). At the fundamental level, the emergence of topological quasi-particles from complicated situations mixing charge and spin-properties will be clarified, both theoretically and experimentally.

Developing an efficient, time-dependent Keldysh formalism for the transport methodology, is warm welcome for this part of the project. It will allow a convenient description of realistic experiments. In addition, the gauge-covariant formalism allows to describe different experiments using a few concepts like the gauge-field. For instance, instead of making different calculations for different spin-orbit symmetries (among the most known ones, one can cite Rashba or Dresselhaus for instance), one can establish some generic formulas for the gauge-potentials, independent of the specific form of the interaction (Konschelle *et al.*, 2015). In addition one can establish easily some connexions between quite different phenomenologies, since the languages of spin and charge will be unified under the banner of the gauge terminology. Finally, note that the generalization of the gauge-covariant transport formalism to the time-dependent Keldysh approach is straightforward, at least at the level of the establishment of the equations. Their resolution will be quite challenging, though. Nevertheless, being a novel method, one can focus at the early stage of the project on simplified setups, before going to more elaborate ones at later stages.

IV. FUNDAMENTAL PROBLEMS IN THEORETICAL PHYSICS

In this section, I propose long term perspective projects. Despite being of more fundamental level, they are also less explored at the present time, and it is thus difficult to

[8] Obviously, the theoretical methods discussed in sections III.A and III.B will help distinguishing the topological sector from the non-topological sector. In this section we discuss an alternative way, of interest for the practical implementations of future topological devices, as well as for the discovery of novel magneto-electric devices.

foreseen whether they will be successful or not. They at least require collaboration with elder scientists in subjects I'm not much acquainted with, like mathematical physicists or more abstract mathematicians for the deformation quantization proposal, or numericians for the semi-classic self-consistent equations. The last proposal dealing with the topological superconductivity in the phase space, is the direct prolongation of the main project presented above.

A. Deformation quantization

The quantization procedure is a long standing problem in mathematical physics. Of course, the so-called canonical quantization is a workable method for physicists, introduced in a more or less ad-hoc way by physicists in the first half of the 20-th century (Dirac, 1982) and culminating with the quantization of the non-Abelian gauge-field (Hooft, 2005). The canonical quantization indeed requires to identify the eigenmodes of the fields, and to quantize them as harmonic oscillators (Dirac, 1982). An other method is to use the Feynman path integral, in which case the classical action is quantized thanks to coherent states representations (Altland and Simons, 2010). None of these methods seem to be really convenient for a mathematical description, despite ongoing efforts. In particular, this is a really sensible problem, since many physicists acknowledge quantum mechanics as the deepest level of understanding, from which the entire world can be understood. Then a clear picture of the mathematical roots of quantum mechanics seem warmly welcome.

In contrary, mathematicians have developed the so-called deformation quantization, when the quantum mechanics is seen as a deformation (with parameter $\hbar$) of the classical mechanics (Sternheimer, 1998). Then the completely workable mathematical structure of the classical mechanics (mainly its symplectic structure inherited from Lagrangian mechanics) can be transposed to the quantum world. In action, the deformation quantization shares common background with the semi-classic methods. In fact, deformation quantization replaces the classical Poisson brackets by a so-called star product, which is nothing but the Moyal one (Moyal, 1947), obtained from the Wigner transformation (Wigner, 1932). In addition, there are deep mathematical proof that the first quantization problem can be render as satisfactory as possible using the methods of deformation quantization (Kontsevich, 2003).

The problem of the second quantization (in particular the quantization of gauge-fields) using the deformation quantization is actually under scrutiny. At a modest physicist level, one might try to help the community of mathematicians in establishing such a tool. In particular, the way physicist envision the semi-classic problem is to first quantize the problem using e.g. the canonical quantization, and then to expand it at the quasi-classic level using the Moyal product expansion (as I actually did in (Konschelle, 2014)). It might be of interest for both the communities of mathematicians and physicists to explore the quasi-classic level without first quantizing a classical theory. This will constitute a first important step toward the acceptation of the deformation quantization methods to the community of physicists, and might guide mathematicians in their quest for alternative methods of quantization.

B. A theory of everything at the quasi-classic level

At the moment the quasi-classic transport formalism suppose the gauge-fields are classical (Konschelle, 2014). Only the electrons are treated quantum mechanically. As such the gauge-fields are always considered as some reservoirs into which the electronic systems can pump some energy, but which can not be influenced back by the fermionic excitations.

It is desirable to complete the interaction between the electrons and the gauge degree of freedom, and to treat the interaction between the fermions and the gauge-bosons in a self-consistent way, when the back-action from the electrons to the fields are included. This would allow to treat many more situations, especially at the smaller scale.

For the purpose of the explanation, let us suppose an heterostructure composed of superconductor and magnetic devices. It has been prove that, under quite generic criteria, a Josephson sandwich made of two superconductors and a spin-textured device would generate a spontaneous current, the so-called φ_0-Josephson effect (Konschelle *et al.*, 2015). In this study, the spontaneous current was generated via a subtle conversion between charge and spin current, and no thermodynamic concept was violated since the super-current propagate without resistance. Nevertheless, the anomalous phase-shift φ_0 was maintained only because the interaction between the superconductor and magnetic orders were not taken into account. In a realistic situation – especially when the magnetic and superconducting devices are quite small – one might wonder whether the phase-shift will survive and generate a current, or whether the current will destroy the magnetic order. Such a line of reasoning was initiated in (Konschelle and Buzdin, 2009) using classical arguments.

I propose in this long-term perspective to study quantum system when action and back-action are important, for instance at a small scale. It requires to treat the gauge degree of freedom as a dynamical variable in the equation of motion in the same manner as the electronic structure is described (see e.g. (Javanainen *et al.*, 1987; Serimaa *et al.*, 1986) for the Abelian case, i.e. the photon gauge field). Despite the equations of motion do not present any difficulty of establishment, the study of these highly non-linear, self-consistent equations would require specific studies, far beyond the scope of my present skills. Nevertheless, at a long term perspective, and once established some collaborations (especially with numericians) one will be able to consider such a problem, of crucial interest for mesoscopic systems.

C. Topological superconductivity in the phase space

In the main project, I already proposed to discuss the real space and the momentum space topology associated to the Majorana modes in superconductors at the semi-classical level The next step will be to describe the topology associated to the phase space completely. This would require a bit of knowledge on symplectic geometry, in addition to some familiarity with the topological aspects in both the real and the momentum spaces. So this part of the project is of middle/long term perspective. I would like to note here that some progress have been done, in a formalism which was not using the principle of gauge covariance (Volovik, 2003).

Especially, this part of the project – once the main project is accomplished – will constitute an important step toward the understanding of more exotic topological degree of freedom in superconductors, as e.g. required for a topological quantum computation (Alicea and Stern, 2014). In particular, the mixing of quantum Hall and superconductivity in heterostructure seems to be the obvious next step once one understands the Majorana modes. This combination is predicted to host non-Abelian anyons, in particular the so-called Fibonacci anyons (Alicea and Stern, 2014).

V. IMPACT OF THE PROJECT

The project is about fundamental physics, and will focus on theoretical methods and concepts, developing new methods and versatile tools to deal with mesoscopic devices allowing coherent manipulation of the quantum states. The project will develop new perspectives in the field of topological superconductivity, which are vividly studied at the moment, using alternative approach than the mainstream one. This original project may progress in slower pace than using more immmediate formalism and methods, but the developments of a transport formalism permit to study much more systems than a wave-function approach, namely impurities, temperature, disordered heterostructures will be discussed. In addition, thermal effect, topological quantum field theory, and magneto-electric phenomenology can be touch upon using the transport formalism, a task of considerable difficulties using the standard Bogoliubov-deGennes approach. With this project, one thus aims at first establishing the transport formalism as a workable method, usefull to everyone in order to predict accurate response of exotic states of matter.

A large part of the motivation is about developing collaborations between theoretical and experimental physics. As such I will provide versatile tools and method able to accurately deal with the technical specificities of realistic experiments, as the inclusion of impurities and defects in heterostructures. One expects to predict new effect in heterostructures mixing superconductivity with spin and charge phenomenologies.

In addition, at a longer time perspective, I aim at developing some versatile (eventually numerical) implementation of quantum equation able to describe in an efficient way superconducting heterostructures. This tool will be of great interest for the development of quantum technologies, when condensed matter comes into fruition in quantum information devices.

VI. CONCLUSION

This project aims first at a deep understanding of the condition of emergence of the neutral Majorana mode in superconducting heterostructures when spin-orbit and spin-splitting effects combine. The simultaneous elaboration of a convenient method for the emergence of a topological quantum field theory, the understanding of the main phenomenology of Majorana modes, namely the thermal Hall effect and its quantum counterpart, and the understanding of the phenomenology associated to the superconducting heterostructures beyond the topological sector – expected to be some kind of magneto-electric phenomenology – will provide a clear and complete description of this elusive topological degree of freedom. This long road project can be realized in a convenient way using the actually developed transport formalism and its future developments, especially towards non-trivial topology of the excitation spectrum of superconducting heterostructures.

References

Abrikosov, A. A., and L. P. Gor'kov (1959), Sov. Phys. JETP **9**, 220.

Abrikosov, A. A., and L. P. Gor'kov (1961), Soviet Physics JETP **12** (6), 1243 .

Abrikosov, A. A., L. P. Gor'kov, and I. E. Dzyaloshinsky (1963), *Methods of quantum field theory in statistical physics* (Prentice Hall).

Aitchison, I. J. R., and A. J. G. Hey (2004), *Gauge Theories in Particle Physics (First volume: From relativistic quantum mechanics to QED ; Second volume: QCD and the electroweak theory)*, 3rd ed. (Taylor and Francis).

Akkermans, É., and G. Montambaux (2004), *Physique m{é}soscopique des {é}lectrons et des photons* (CNRS {É}ditions - EDP Sciences).

Alicea, J. (2012), Reports on progress in physics. Physical Society (Great Britain) **75** (7), 076501, arXiv:1202.1293 .

Alicea, J., Y. Oreg, G. Refael, F. von Oppen, and M. P. A. Fisher (2011), Nature Physics **7** (5), 412, arXiv:1006.4395 .

Alicea, J., and A. Stern (2014), arXiv:1410.0359 .

Altland, A., and B. Simons (2010), *Condensed matter field theory* (Cambridge University Press).

Beenakker, C. (2013), Annual Review of Condensed Matter Physics **4** (1), 113, arXiv:1112.1950 .

Berche, B., and E. Medina (2013), European Journal of Physics **34** (1), 161, arXiv:1210.0105 .

Bergeret, F. S., A. F. Volkov, and K. B. Efetov (2005), Reviews of Modern Physics **77** (4), 1321.

Buzdin, A. I. (2005), Reviews of Modern Physics **77** (3), 935.

De Gennes, P.-G. (1999), *Superconductivity of metals and alloys* (Advanced Book Classics, Westview Press).

Deng, M. T., C. L. Yu, G. Y. Huang, M. Larsson, P. Caroff, and H. Q. Xu (2012), arXiv , 7arXiv:1204.4130 .

Dirac, P. A. M. (1982), *The Principles of Quantum Mechanics*, 4th ed. (Oxford University Press, USA).

Dunne, G. V. (1998), in *Topological Aspects of Low Dimensional Systems (Lectures at the 1998 Les Houches Summer School)*, i (Springer) p. 76, arXiv:9902115 [hep-th] .

Eilenberger, G. (1968), Zeitschrift für Physik **214** (2), 195.

Elze, H.-T., and U. Heinz (1989), Physics Reports **183** (3), 81.

Eschrig, M. (2015), arXiv:1509.02242 .

Fröhlich, J., and U. M. Studer (1992), Communications in Mathematical Physics **148** (3), 553.

Fu, L., and C. L. Kane (2008), Physical Review Letters **100** (9), 096407, arXiv:0707.1692 .

Giazotto, F., T. T. Heikkilä, A. Luukanen, A. M. Savin, and J. P. Pekola (2006), Reviews of Modern Physics **78** (1), 217, arXiv:0508093 [cond-mat] .

Gorini, C., P. Schwab, R. Raimondi, and A. L. Shelankov (2010), Physical Review B **82** (19), 195316, arXiv:1003.5763 .

Gor'kov, L. P. (1958), Sov. Phys. JETP **7**, 505.

Gor'kov, L. P. (1959), Sov. Phys. JETP **9**, 1364.

Gor'kov, L. P., and E. Rashba (2001), Physical Review Letters **87** (3), 37004.

Hasan, M. Z., and C. L. Kane (2010), Reviews of Modern Physics **82** (4), 3045.

Hooft, G. t. (2005), *50 Years Of Yang-Mills Theory*, edited by G. t. Hooft (World Scientific Publishing Co. Pte. Ltd.).

Javanainen, J., S. Varró, and O. Serimaa (1987), Physical Review A **35** (7), 2791.

Kadanoff, L. P., and G. Baym (1962), *Quantum Statistical Mechanics*, edited by D. Pines (Westview Press).

Kitaev, A. (2001), Physics-Uspekhi **44** (10S), 131, arXiv:0010440 [cond-mat] .

Kitaev, A. (2003), Annals of Physics **303** (1), 2, arXiv:9707021 [quant-ph] .

Konschelle, F. (2014), The European Physical Journal B **87** (5), 119, arXiv:1403.1797 .

Konschelle, F., and Y. M. Blanter (2015), Europhysics Letters (EPL) **109**, 68004, arXiv:1110.1357 .

Konschelle, F., and A. I. Buzdin (2009), Physical Review Letters **102** (1), 17001, arXiv:0810.4286 .

Konschelle, F., I. V. Tokatly, and F. S. Bergeret (2015), Physical Review B **92** (12), 125443, arXiv:1506.02977 .

Kontsevich, M. (2003), Letters in Mathematical Physics **66** (3), 157, arXiv:9709040 [q-alg] .

Kopnin, N. B. (2001), *Theory of nonequilibrium superconductivity* (Oxford {U}niversity {P}ress, Oxford).

Langenberg, D. N., and A. I. Larkin (1986), *Nonequilibrium superconductivity* (North-Holland, Amsterdam).

Larkin, A. I., and Y. N. Ovchinnikov (1969), Sov. Phys. JETP **28**, 1200.

Leggett, A. J. (1975), Reviews of Modern Physics **47** (2), 331.

Linder, J., and J. W. A. Robinson (2015), Nature Physics **11** (4), 307, arXiv:1510.00713 .

London, F. (1961), *Superfluids, volume I: Macroscopic theory of superconductivity*, 2nd ed. (Dover Publications, Inc.).

Luttinger, J. (1964), Physical Review **135** (6A), A1505.

Majorana, E. (1937), Il Nuovo Cimento **14** (4), 171.

Mourik, V., K. Zuo, S. M. Frolov, S. R. Plissard, E. P. a. M. Bakkers, and L. P. Kouwenhoven (2012), Science (New York, N.Y.) **336** (6084), 1003.

Moyal, J. E. (1947), Mathematical Proceedings of the Cambridge Philosophical Society **45** (01), 99.

Nadj-Perge, S., I. K. Drozdov, B. a. Bernevig, and A. Yazdani (2013), Physical Review B **88** (2), 020407, arXiv:1303.6363 .

Nadj-Perge, S., I. K. Drozdov, J. Li, H. Chen, S. Jeon, J. Seo, A. H. MacDonald, B. A. Bernevig, and A. Yazdani (2014), Science **346** (6209), 602, arXiv:1410.0682 .

Nayak, C., A. Stern, M. H. Freedman, and S. Das Sarma (2008), Reviews of Modern Physics **80** (3), 1083, arXiv:0707.1889 .

Nazarov, Y. V., and Y. M. Blanter (2009), *Quantum transport: introduction to nanoscience* (Cambridge University Press).

Nielsen, M. A., and I. L. Chuang (2000), *Quantum Computation and Quantum Information*, edited by C. U. Press.

Polkovnikov, A. (2010), Annals of Physics **325** (8), 1790, arXiv:0905.3384 .

Qi, X.-L. (2013), in *Topological Insulators*, edited by L. W. Molenkamp and M. Franz, 1st ed., Chap. 4 (Elsevier).

Qi, X.-l., and S.-c. Zhang (2011), Reviews of Modern Physics **83** (4), 1057, arXiv:1008.2026 .

Rammer, J., and H. Smith (1986), Reviews of Modern Physics **58** (2), 323.

Read, N., and D. Green (2000), Physical Review B **61** (15), 10267, arXiv:9906453 [cond-mat] .

Serimaa, O., J. Javanainen, and S. Varró (1986), Physical Review A **33** (5), 2913.

Son, D. T., and N. Yamamoto (2012), Physical Review Letters **109** (18), 181602, arXiv:1203.2697 .

Son, D. T., and N. Yamamoto (2013), Physical Review D **87** (8), 085016, arXiv:1210.8158 .

Stern, A., and N. H. Lindner (2013), Science **339** (6124), 1179.

Sternheimer, D. (1998), pp. 107–145, arXiv:9809056 [math] .

Sundaram, G., and Q. Niu (1999), Physical Review B **59** (23), 14915, arXiv:9908003 [cond-mat] .

Szombati, D. B., S. Nadj-Perge, D. Car, S. R. Plissard, E. P. A. M. Bakkers, and L. P. Kouwenhoven (2015), arXiv:1512.01234 .

Volovik, G. E. (1988), Physica Scripta **38** (3), 321.

Volovik, G. E. (2003), *Universe in a Helium Droplet* (Oxford University Press, Oxford).

Volovik, G. E., and V. M. Yakovenko (1989), Journal of Physics: Condensed Matter **1** (31), 5263.

Wickles, C., and W. Belzig (2013), Physical Review B **88** (4), 045308, arXiv:1209.4933 .

Wigner, E. P. (1932), Physical Review **40** (5), 749.

Wilhelm, F. K., W. Belzig, C. Bruder, G. Schön, and A. D. Zaikin (1999), Superlattices and Microstructures **25** (5-6), 1251, arXiv:9812297 [cond-mat] .

Wong, C. H., and Y. Tserkovnyak (2011), Physical Review B **84** (11), 115209, arXiv:1102.1121 .

Xiao, D., M.-C. Chang, and Q. Niu (2009), Reviews of Modern Physics **82** (3), 48, arXiv:0907.2021 .